Finite Element Formulations

for Statics and Dynamics of Plane Structures (with Matlab)

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Dedication

To our families and students...
to the future!

Preface

Finite Element Method (FEM) has become a compulsory knowledge for present day engineers as it allows (what used to be) very complex behavior of physical phenomenon to be known (approximately) and exploited. However, the teaching and learning of the subject are still difficult, as usually described by the learners. In the authors' opinion, the difficulties can be blamed on the fragmentation (of the discussions) between mathematics, engineering fundamentals and the basic concepts of numerical method. Realizing this, the authors are promoting a new approach in this book by insisting for a "close-loop" type of discussion in each topic or chapter. A topic always begins with the derivation of the differential equation/s (of the problem). It is followed by the conversion of the equation/s into matrix forms through finite element argument. A worked example is then immediately given (in a very detailed manner) before it is closed by a MATLAB source code.

This book is neither designed to be a complete book on FEM nor intended to dwell on the practice of FEM modelling (using on-shelf software). Instead, it is prepared with a specific idea in mind; the book is about easy tracing of the evolution of the finite element formulation and thus has the following features:

1. A complete loop in each formulation (from the derivation of the partial/ordinary differential equations to the discretization of the equations into matrix system to the computer programming)

2. Increasing complexity from one formulation to another (that is, from bar element to beam element to truss element to frame to free vibration and buckling problems and finally to forced vibration of the structures)

For the above reasons, this book does not have abundant worked examples but focusing on a few examples, detailing every step so as to make obvious what has been discussed in the preceding text and what awaits in succeeding source code. Also (with the specific example per chapter), the evolution and the continuity of arguments can be clearly established from one chapter to another (It is the authors' opinion that too many examples per chapter would make the relationship between examples in different chapters less obvious). Nevertheless, there are plenty of solved exercises provided.

This book has evolved from a series of lecture notes of the first author refined over the period of ten years with the co-authors. It revolves around frame structural analysis, both statics and dynamics. In Chapter 1, the book begins with the basic concepts of numerical methods before introducing the concept of Galerkin weighted residual method towards the end. Chapter 2 focuses on bar finite element. The formulation of beam element is discussed in Chapter 3. Chapter 4 discusses the concept of space orientation and the assembly of elements for plane structures (truss and frame). Chapter 5 discusses two classes of eigenvalue problems; free vibration and buckling of structures. Chapter 6 details the formulation of forced vibration of bar, beam and plane frame. In this final chapter, time discretization by finite difference method is introduced.

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October, 2018
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1 Basic Concept of Numerical Techniques

1.1 Introduction: What is Finite Element Method?

Finite Element Method or FEM can be both "everything" and "nothing". At one end, FEM is everything when it allows engineers to get information (i.e. stresses, displacements, forces) of complex physical phenomenon for design purposes. At the other end, FEM is nothing because the information obtained is actually nothing but a solution to a partial differential equation (PDE) or ordinary differential equations (ODE). In other words, FEM is nothing but another numerical method to solve PDE or ODE.

Realizing how FEM can be "everything" is important as it can motivate the study. But realizing how FEM can be "nothing" is just as important as it can guide the proper learning of FEM that is, any discussion must begin from the first principle (i.e. PDE or ODE) if strong understanding is desired.

To note, since ODE is a special case of PDE, from now on, PDE will be quoted when references to both class of equations are made.

A formal description of FEM can be given as follows. FEM is a numerical method that approximates the solution of a PDE by breaking up the physical domain into smaller elements where adjacent elements are connected at nodes to form a mesh. Such a mesh formation process is technically termed as element assembly. In FEM, the dependent variables at nodal locations (referred as the degree of freedoms) are interpolated by shape functions. Insertion of these interpolation functions into the PDE produces a residual error function which, when forced to zero with the employment of weighted residual method, in turn, produces a matrix system. Imposition of boundary conditions can be done directly before the unknown degree of freedoms be solved.

1.2 Basic Concept of Numerical Techniques

Having said how FEM is just another numerical method, below is the list of established numerical methods.

i. Finite Element Method (FEM)
ii. Finite Difference Method (FDM)
iii. Boundary Element Method (BEM)
iv. Meshless or Meshfree Methods (Meshfree)

However, despite their variations, all the methods share similar concept that is;

> *"to convert the continuous nature of PDE (or ODE) into 'equivalent' simultaneous algebraic equations in the form of a matrix system".*

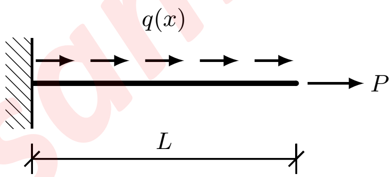

Figure 1.1: Bar element / structure.

In elaborating the concept, we discuss herein the solution of the simplest forms of ODE, that is of a bar element. By leaving the derivation for later, the ODE of a bar element (as shown in Fig. 1.1) can be given as:

Domain equation

$$EA\frac{d^2u}{dx^2} = -q \tag{1.1}$$

2 Galerkin Formulation: Bar Element

2.1 Introduction

In the previous chapter, we have discussed the basic concept of numerical technique and the basic concept of WRM. In this chapter, we are going to discuss on the specific form of WRM that is employed in the present day of FEM that is Galerkin WRM. However, since it has also been mentioned earlier that FEM is nothing but a numerical solution to a PDE, it is important in any FEM endeavour for the analyst to be familiar with the relevant PDE including its derivation and its closed-form solution (if available). This way, when the analyst is required to embark on a new project or study, he or she is already being trained to look at the problem from the first principle, and identify all the relevant aspects or concerns, before he or she employs FEM in getting the solution of the problem.

2.2 Ordinary Differential Equation of Bar Problem

As mentioned, it is vital for an analyst to get into the problem from the first principle and in many cases; this would mean from the derivation of the relevant PDE (or ODE). Since in the previous chapter, we have been introduced to the ODE of bar problem, herein we are going to show the derivation of the ODE. Fig. 2.1(a) shows a bar element subjected to an external distributed load, $q(x)$ and an end load, P and its differential element.

It can be argued that, although we have an axial force, F at the left side

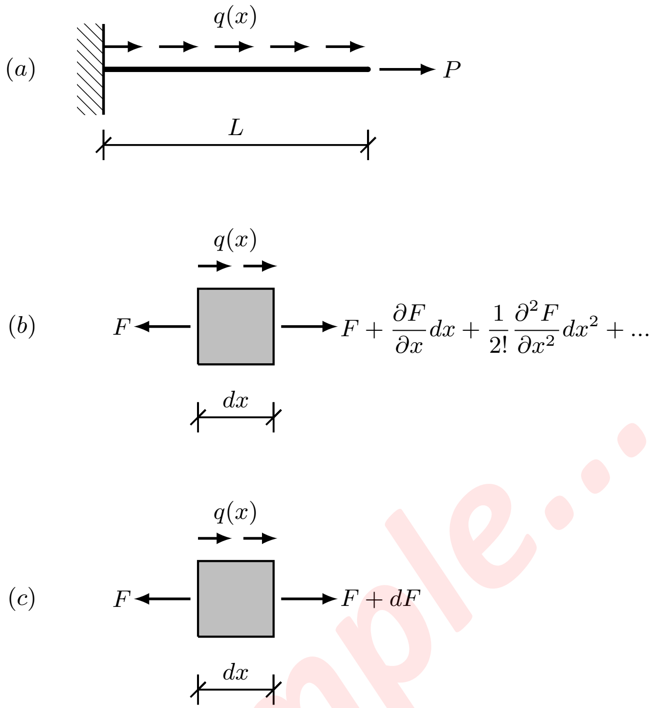

Figure 2.1: Bar structure and its differential element.

of the differential element, due to the 'disturbance' along the differential length, dx (due to external load, for example) the magnitude of the axial force at the right side of the differential element must change. However we don't know the exact change of this force, else we would not have this problem in the first place, would we? But by assuming the change is continuous, we can say that the force at the right-side of the differential element can be represented by Taylor series, as shown in Fig. 2.1(b).

But by assuming higher order terms as insignificant and since this is a 1D problem (i.e. $\partial(\quad) = d(\quad)$), a state as shown in Fig. 2.1(c) is considered. This is an important argument thus must be grasped by the readers because as we going to see later, the derivation of PDE (or ODE) for other problems will be based on the same argument as well.

Having established the differential element and the corresponding forces

3 Galerkin Formulation: Beam Element

3.1 Introduction

In the previous chapter, we have discussed the concept of Galerkin WRM hence FEM and formulated the discretised equation for bar element. Herein, we are going to continue our discussion by formulating the discretised equation for beam problem. We begin by deriving the Euler-Bernoulli differential equation of beam which is given next.

3.2 Ordinary Differential Equation of Euler-Bernoulli Beam

An Euler-Bernoulli beam is a structural member that resists loads by bending and shearing. The corresponding deformation would be rotation and translation. Consider a structural beam which is subjected to a distributed load, $q(x)$ as shown in Fig. 3.1(a). The differential equation for such a beam can be derived for static loading by considering its differential element as shown in Fig. 3.1(b). Also shown is the typical arrangement of a beam structure.

As argued previously for bar element (in Chapter 2), although we have moment force, M and shear force, V at the left side of the differential element, due to the 'disturbance' along the differential length, dx (due to external load, for example) the magnitude of these forces at the right side of the differential element must change. However we don't know the exact change of these forces, else we would not have the problem in the first place would we? But by assuming the change is continuous, we can say that the forces at the right-side of the differential element can be represented by a Taylor series, as shown in Fig. 3.1(b).

By assuming higher order terms as insignificant and since this is a 1D problem (i.e.$\partial(\quad) = d(\quad)$), a state as shown in Fig. 3.1(c) is considered. This is an important argument thus must be grasped by the readers because as we are going to see later, the derivation of PDE (or ODE) for other problems will be based on the same argument as well.

Having established the differential element and the corresponding forces acting on it, we are in the position to derive the ODE for beam bar problem. However, it must be noted again that the following differential equation does not consider axial deformation thus the absent of axial forces. Beams allowing such forces are called beam-column, of which the FEM formulation is given in the next chapter. Also, present formulation assumes slope is equalled to rotation. A more general formulation would be the Timoshenko beam theory as it allows different values for the two entities, but it is not included in our discussion.

Based on Fig. 3.1(c), the following equilibrium of forces can be employed:

$$\sum F_x = 0 \tag{3.1}$$

$$\sum F_y = 0 \tag{3.2}$$

$$\sum M_z = 0 \tag{3.3}$$

which yield

$$V - (V - dV) - q\,dx = 0 \tag{3.4}$$

$$-M + (M + dM) - V\,dx - q\frac{dx^2}{2} = 0 \tag{3.5}$$

where q is the distributed transverse external loading and w the deflection of the beam.

4 Plane Structures: Truss and Frame

4.1 Introduction

In the previous chapters, we have discussed the derivation of FEM formulation for bar and beam elements. However, these elements were arranged and assembled in a line. A more general arrangement would require the elements to be arbitrarily oriented and assembled. Assembly of such oriented elements would make up a truss system and a frame system, respectively. As we are going to see, the orientation process requires the introduction of the transformation matrix. The use of this matrix is to transform local entities into global entities. Another point to emphasize is the introduction of two degree of freedoms and two load components into the bar element's global representation. Also, since the construction of a frame would require the transfer of axial load/force, a beam element is supplied with extra degree of freedoms in the axial direction. As can be seen, this will involve the combination of previously derived bar and beam elements to form what is called beam-column element.

4.2 Truss System

A truss system is an assembly of inclined bar elements. Figure 4.1 shows a typical arrangement of a plane truss system.

To allow for the inclination of the truss members and the corresponding assembly, a bar element formulation must be re-expressed in a global manner. In a global axis, a bar element would have two degree of freedoms per node. Such a transformation requires us to establish what is known as a

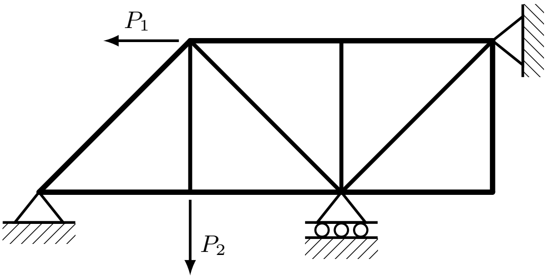

Figure 4.1: A typical truss system.

transformation matrix as discussed next.

4.2.1 Bar Transformation Matrix

Consider an inclined bar as shown in Fig. 4.2, together with the newly introduced elemental global direction dofs and previously defined local dofs. Note that, to distinguish between local and elemental global direction dofs, the former is primed.

For a linear bar (two-nodes bar), by considering the geometry of Fig. 4.2 the relationship between the local and the elemental global dofs can be given as:

$$u'_1 = u_1 \cos\beta + u_2 \sin\beta \tag{4.1a}$$

$$u'_2 = u_3 \cos\beta + u_4 \sin\beta \tag{4.1b}$$

In matrix forms, Eq. (4.1a) can be given as:

$$\begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta & 0 & 0 \\ 0 & 0 & \cos\beta & \sin\beta \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \tag{4.2}$$

5 Eigenproblems: Free Vibration and Buckling

5.1 Introduction

In previous chapters, we have been dealing with structures which are loaded in the direction of the degree of freedoms; point loads (either equivalent or nodal loads) in the direction of the translational dof or applied moment in the direction of the rotational dof. Herein, we are going to discuss a quite different situation that is, the deformation which is 'ungoverned' by loads that are acting on the structure.

5.2 "Ungoverned by the Loads" and Eigenproblems

The word "ungoverned", however, requires further elaboration. The word refers to the statement that the load vector, $\{r\}$ or $\{R\}$ is set to zero. Mathematically this means solving only the homogenous part of the differential equation of the problem. The resulting values would be some properties of the structure and their corresponding deformation.

For example, free vibration refers to the "vibration" of a structure which is described by the natural frequencies and the corresponding mode shapes, without any explicit consideration on the type of external loading. Only during the discussion of resonance, these frequencies would then be compared with the incoming frequencies (external loads).

Same goes to the discussion of buckling of a structure. The load that would cause the buckling will be specific critical values of compressive axial force termed as buckling loads. As will be seen, since Euler-Bernoulli beam formulation able to capture such a phenomenon and determine the buckling loads and their corresponding buckling modes without the need to introduce axial dofs, it should be obvious that the bucking loads and their modes are not governed by the external applied loads.

Since both values (natural frequencies, buckling loads) are not governed by the loads, they must be some properties of the structure hence the name "eigen" which means "inherent" or "characteristic" in German. Then, what governs their values? Something must affect their values, must not they? As some properties, they are governed by other properties of the structures; material and geometrical properties. In our discussion on bar, beam and their inclined elements, these would be Young's modulus, E, cross-sectional area, A, second moment of area, I and element's length, L.

Also, since these values are "ungoverned" by the loads, we will see that the FEM formulation for both problems will involve with the discretization of their differential equations without the forcing terms hence the setting up of load vector, $\{r\}$ or $\{R\}$ to zero.

Accordingly, all physical problems that fall under the same argument are called eigenproblems as their equilibrium equations can all be arranged into a standard mathematical statement. If $[A]$ and $[B]$ are square matrices with known coefficients and λ is an unknown constant, an eigenproblem is a problem that can be described by the following typical mathematical statement:

$$([A] + \lambda[B])\{d\} = 0 \tag{5.1}$$

where λ is termed as eigenvalue and $\{d\}$ is termed as eigenvector.

For free vibration analysis, matrix $[A]$ represents the stiffness matrix, $[K]$, matrix $[B]$ represents the mass matrix, $[M]$, constant λ represents the square of natural frequencies and vector $\{d\}$ represents the vector of mode shape dofs $\{\hat{d}\}$.

For buckling problem, matrix $[A]$ represents the stiffness matrix, $[K]$, matrix $[B]$ represents the stress stiffness matrix $[K_G]$, constant λ represents